

# Values of the Pukánszky invariant in McDuff factors

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## Abstract

In 1960 Pukánszky introduced an invariant associating to every masa in a separable  $\text{II}_1$  factor a non-empty subset of  $\mathbb{N} \cup \{\infty\}$ . This invariant examines the multiplicity structure of the von Neumann algebra generated by the left-right action of the masa. In this paper it is shown that any non-empty subset of  $\mathbb{N} \cup \{\infty\}$  arises as the Pukánszky invariant of some masa in a separable McDuff  $\text{II}_1$  factor containing a masa with Pukánszky invariant  $\{1\}$ . In particular the hyperfinite  $\text{II}_1$  factor and all separable McDuff  $\text{II}_1$  factors with a Cartan masa satisfy this hypothesis. In a general separable McDuff  $\text{II}_1$  factor we show that every subset of  $\mathbb{N} \cup \{\infty\}$  containing  $\infty$  is obtained as a Pukánszky invariant of some masa.

## 1 Introduction

In [12] Pukánszky introduced an invariant for a maximal abelian self-adjoint subalgebra (masa) inside a separable  $\text{II}_1$  factor, which he used to exhibit a countable infinite family of singular masas in the hyperfinite  $\text{II}_1$  factor no pair of which are conjugate by an automorphism. The invariant associates a non-empty subset of  $\mathbb{N} \cup \{\infty\}$  to each masa  $A$  in a separable  $\text{II}_1$  factor  $N$  as follows. Let  $\mathcal{A}$  be the abelian von Neumann subalgebra of  $\mathbb{B}(L^2(N))$  generated by  $A$  and  $JAJ$ , where  $J$  denotes the canonical involution operator on  $L^2(N)$ . The orthogonal projection  $e_A$  from  $L^2(N)$  onto  $L^2(A)$  lies in  $\mathcal{A}$  and the algebra  $\mathcal{A}'(1 - e_A)$  is type I so decomposes as a direct sum of type  $\text{I}_n$ -algebras. The Pukánszky invariant of  $A$  is the set of those  $n \in \mathbb{N} \cup \{\infty\}$

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appearing in this decomposition and is denoted  $\text{Puk}(A)$ . See also [14, Section 2].

There has been recent interest in the range of values of the Pukánszky invariant in various  $\text{II}_1$  factors. Nesheveyev and Størmer used ergodic constructions to show that any set containing 1 arises as a Pukánszky invariant of a masa in the hyperfinite  $\text{II}_1$  factor [7, Corollary 3.3]. Sinclair and Smith produced further subsets using group theoretic properties in [14] and with Dykema in [4], which also examines free group factors. In the other direction Dykema has shown that  $\sup \text{Puk}(A) = \infty$ , whenever  $A$  is a masa in a free group factor [3].

In this paper we show that every non-empty subset of  $\mathbb{N} \cup \{\infty\}$  arises as the Pukánszky invariant of a masa in the hyperfinite  $\text{II}_1$  factor by means of an approximation argument. More generally we obtain the same result in any separable McDuff  $\text{II}_1$  factor containing a simple masa, that is one with Pukánszky invariant  $\{1\}$  (Corollary 6.2). These factors are the first for which the range of the Pukánszky invariant has been fully determined. Without assuming the presence of a simple masa we are able to show that every separable McDuff  $\text{II}_1$  factor contains a masa with Pukánszky invariant  $\{\infty\}$  and hence we obtain every subset of  $\mathbb{N} \cup \{\infty\}$  containing  $\infty$  as a Pukánszky invariant of some masa in these factors (Theorem 6.7). In particular, there are uncountably many singular masas in any separable McDuff factor, no pair of which is conjugate by an automorphism of the factor.

Section 4 contains a construction for producing masas in McDuff  $\text{II}_1$  factors. Given a McDuff  $\text{II}_1$  factor  $N_0$  we shall repeatedly tensor on copies of the hyperfinite  $\text{II}_1$  factor — this gives us a chain  $(N_s)_{s=0}^\infty$  of  $\text{II}_1$  factors whose direct limit  $N$  is isomorphic to  $N_0$ . We shall produce a masa  $A$  in  $N$  by giving an approximating sequence of masas  $A_s$  in each  $N_s$  such that  $A_s \subset A_{s+1}$  and defining  $A = (\bigcup_{s=0}^\infty A_s)''$ . This idea has its origin in [16] working in the hyperfinite  $\text{II}_1$  factor arising as the infinite tensor produce of finite matrix algebras, although using finite matrix algebras can only yield masas with Pukánszky invariant  $\{1\}$ , [17, Theorem 4.1].

In the remainder of the introduction we outline the construction of a masa with Pukánszky invariant  $\{2, 3\}$ . Initially we shall produce a masa  $A_1$  in  $N_1$  such that the multiplicity structure of  $\mathcal{A}_1$  (the algebra generated by the left-right action of  $A_1$  on  $L^2(N_1)$ ) is represented by Figure 1. By this we mean that  $e$  is a projection of trace  $1/2$  in  $A$  and that  $\mathcal{A}'_1 e J e J$  and  $\mathcal{A}'_1 e^\perp J e^\perp J$  are both type  $\text{I}_1$ , while  $\mathcal{A}'_1 e J e^\perp J$  and  $\mathcal{A}'_1 e^\perp J e J$  are type  $\text{I}_2$ .

At the second stage we subdivide  $e$  and  $e^\perp$  to obtain four projections in

	$e$	$e^\perp$
$e$	1	2
$e^\perp$	2	1

Figure 1: Symbolic description of the multiplicity structure of  $\mathcal{A}_1$ .

$A_2$  and arrange for the multiplicity structure of  $\mathcal{A}_2$  to be represented by the left diagram in Figure 2. We then cut each of these projections in half again and ensure that the multiplicity structure of  $\mathcal{A}_3$  is represented by the second diagram in Figure 2, where 1's appear down the diagonal. It is important to do this in such a way that a limiting argument can be used to obtain the multiplicity structure of  $\mathcal{A} = (A \cup JAJ)''$ . If this is done successfully, then the multiplicity structure of  $\mathcal{A}$  will be represented by Figure 3, where the diagonal line has multiplicity 1. If we further ensure that the projections

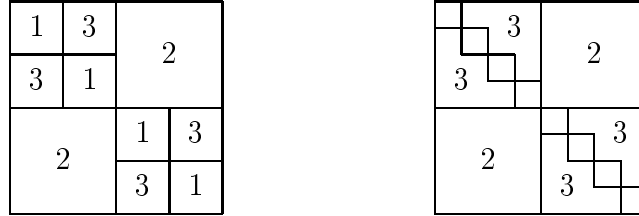


Figure 2: The multiplicity structures of  $\mathcal{A}_2$  and  $\mathcal{A}_3$ .

used to cut down the masas  $A_r$  in this construction generate  $A$ , then the diagonal line in Figure 3 corresponds to the projection  $e_A$  with range  $L^2(A)$  and this is the projection explicitly removed in the definition of  $\text{Puk}(A)$ . The resulting masa  $A$  will then have Pukánszky invariant  $\{2, 3\}$  as required.

To get from Figure 1 to the left diagram in Figure 2 in a compatible way, we ‘tensor on’ the diagram in Figure 4. This is done by producing masas  $D_1, D_2, D_3, D_4$  in the hyperfinite  $\text{II}_1$  factor  $R$  such that  $(D_i \cup JD_j J)'$  is type  $\text{I}_1$  unless  $i, j$  is the unordered pair  $\{1, 2\}$  or  $\{3, 4\}$ . In these cases  $(D_i \cup JD_j J)'$  is type  $\text{I}_3$ . Given projections  $e_1, e_2, e_3, e_4$  in  $A_1$  with  $e = e_1 + e_2$  and  $e^\perp = e_3 + e_4$

3	3
2	3

Figure 3: The multiplicity structure of  $\mathcal{A}$ .

and  $\text{tr}(e_i) = 1/4$  for each  $i$  we shall define  $A_2$  in  $N_2 = N_1 \overline{\otimes} R$  by

$$A_2 = \bigoplus_{i=1}^4 A_1 e_i \overline{\otimes} D_i.$$

In this way  $\mathcal{A}_2$  has the required multiplicity structure.

1	3	1	1
3	1	1	1
1	1	1	3
1	1	3	1

Figure 4: Mixed Pukánszky invariant structure of the masas  $D_1, D_2, D_3, D_4$ .

In sections 2 and 3 we develop the concept of mixed Pukánszky invariants of pairs of masas to handle the families  $(D_i)$ , which we will repeatedly adjoin. The main result is Theorem 3.5, which ensures that the family  $D_1, D_2, D_3, D_4$  above, and other families in this style can indeed be found. In section 4 we give the details of the inductive construction and in section 5 we compute the Pukánszky invariant of the resulting masa. We end in section 6 by collecting together the main results.

## 2 Mixed Pukánszky Invariants

In this paper all  $\text{II}_1$  factors will be separable. In this way we only need one infinite cardinal denoted  $\infty$ . We shall write  $\mathbb{N}_\infty$  for the set  $\mathbb{N} \cup \{\infty\}$  henceforth.

**Definition 2.1.** Given a type I von Neumann algebra  $M$  we shall write  $\text{Type}(M)$  for the set of those  $m \in \mathbb{N}_\infty$  such that  $M$  has a non-zero component of type  $I_m$ .

Given a  $II_1$  factor  $N$ , write  $\text{tr}$  for the unique faithful trace on  $N$  with  $\text{tr}(1) = 1$ . For  $x \in N$ , let  $\|x\|_2 = \text{tr}(x^*x)^{1/2}$ , a pre-Hilbert space norm on  $N$ . The completion of  $N$  in this norm is denoted  $L^2(N)$ . Define a conjugate linear isometry  $J$  from  $L^2(N)$  into itself by extending  $x \mapsto x^*$  by continuity from  $N$ .

**Definition 2.2.** Given two masas  $A$  and  $B$  in a  $II_1$  factor  $N$  define the *mixed Pukánszky invariant* of  $A$  and  $B$  to be the set  $\text{Type}((A \cup JBJ)')$ , where the commutant is taken in  $\mathbb{B}(L^2(N))$ . We denote this set  $\text{Puk}(A, B)$  or  $\text{Puk}_N(A, B)$  when it is necessary. Note that  $\text{Puk}(A, A) = \text{Puk}(A) \cup \{1\}$  for any masa  $A$ , the extra 1 arising as the Jones projection  $e_A$  is not removed in the definition of  $\text{Puk}(A, A)$ .

It is immediate that  $\text{Puk}(A, B)$  is a conjugacy invariant of a pair of masas  $(A, B)$  in a  $II_1$  factor, i.e. that if  $\theta$  is an automorphism of  $N$  we have  $\text{Puk}(A, B) = \text{Puk}(\theta(A), \theta(B))$ . If we only apply  $\theta$  to one masa in the pair then we may get different mixed invariants. For an inner automorphism this is not the case.

**Proposition 2.3.** *Let  $A$  and  $B$  be masas in a  $II_1$  factor  $N$ . For any unitaries  $u, v \in N$  we have*

$$\text{Puk}(uAu^*, vBv^*) = \text{Puk}(A, B).$$

*Proof.* Consider the automorphism  $\Theta = \text{Ad}(uJvJ)$  of  $\mathbb{B}(L^2(N))$ , which has  $\Theta(A) = uAu^*$  and  $\Theta(JBJ) = JvBv^*J$ . Therefore  $(A \cup JBJ)'$  and  $(uAu^* \cup J(vBv^*)J)'$  are isomorphic, so have the same type decomposition.  $\square \quad \square$

The Pukánszky invariant is well behaved with respect to tensor products [14, Lemma 2.1]. So too is the mixed Pukánszky invariant. Given  $E, F \subset \mathbb{N}_\infty$  write  $E \cdot F = \{mn \mid m \in E, n \in F\}$ , where by convention  $n\infty = \infty n = \infty$  for any  $n \in \mathbb{N}_\infty$ .

**Lemma 2.4.** *Let  $(N_i)_{i \in I}$  be a countable family of finite factors. Suppose that we have masas  $A_i$  and  $B_i$  in  $N_i$  for each  $i \in I$ . Let  $N$  be the finite factor obtained as the infinite von Neumann tensor product of the  $N_i$  with respect*

to the product trace and let  $A$  and  $B$  be the infinite tensor products of the  $A_i$  and  $B_i$  respectively. Then  $A$  and  $B$  are masas in  $N$ . When  $I$  is finite,

$$\text{Puk}_N(A, B) = \prod_{i \in I} \text{Puk}_{N_i}(A_i, B_i).$$

If  $I$  is infinite, and each  $\text{Puk}_{N_i}(A_i, B_i) = \{n_i\}$  for some  $n_i \in \mathbb{N}_\infty$ , then  $\text{Puk}_N(A, B) = \{n\}$ , where  $n = \prod_I n_i$ , when all but finitely many  $n_i = 1$ , and  $n = \infty$  otherwise.

*Proof.* That  $A$  and  $B$  are masas follows from Tomita's commutation theorem, see [6, Theorem 11.2.16]. Suppose first that  $I$  is finite. For each  $i \in I$ , let  $(p_{i,n})_{n \in \mathbb{N}_\infty}$  be the decomposition of the identity projection into projections in  $(A_i \cup JB_iJ)'' \subset B(L^2(N_i))$  such that  $(A_i \cup JB_iJ)'p_{i,n}$  is type  $I_n$  for each  $n \in \mathbb{N}_\infty$  (some of these projections may be zero). Then given any family  $(n_i)_i$  in  $\mathbb{N}_\infty$ ,  $p = \bigotimes_{i \in I} p_{i,n_i}$  is a central projection in  $(A \cup JBJ)'$  and  $(A \cup JBJ)'p$  is type  $I_m$  where  $m = \prod_{i \in I} n_i$ . All these projections are mutually orthogonal with sum 1. Therefore  $\text{Puk}_N(A, B)$  consists of those  $m$  such that  $p \neq 0$  and this occurs if and only if all the corresponding  $p_{i,n_i}$  appearing in the tensor product are non-zero. These are precisely the  $m$  in  $\prod_{i \in I} \text{Puk}_{N_i}(A_i, B_i)$ .

Suppose  $I$  is infinite and each  $\text{Puk}_{N_i}(A_i, B_i) = \{n_i\}$ , for some  $n_i \in \mathbb{N}_\infty$ . Let  $\mathcal{A}_i = (A_i \cup JB_iJ)'' \subset B(L^2(N_i))$  and  $\mathcal{A}'_i$  its commutant of  $\mathcal{A}_i$  in  $B(L^2(N_i))$ . Let  $\mathcal{A} = (A \cup JBJ)''$  in  $B(L^2(N))$  and  $\mathcal{A}'$  the commutant of  $\mathcal{A}$  in this algebra. The Tomita commutation theorem gives

$$\mathcal{A}' = \overline{\bigotimes \mathcal{A}'_i} \subseteq \overline{\bigotimes B(L^2(N_i))} \cong B(L^2(N)).$$

Since each  $\mathcal{A}'_i \cong \mathcal{A}_i \overline{\otimes} \mathbb{M}_{n_i}$ , where  $\mathbb{M}_{n_i}$  is the  $n_i \times n_i$  matrices (or  $B(H)$  for some separable infinite dimensional Hilbert space when  $n_i = \infty$ ). Thus

$$\mathcal{A}' \cong \left( \overline{\bigotimes A_i} \right) \overline{\otimes} \left( \overline{\bigotimes \mathbb{M}_{n_i}} \right) \cong A \overline{\otimes} \mathbb{M}_n,$$

so  $\mathcal{A}'$  is homogenous of type  $I_n$ . □ □

Given two masas  $A$  and  $B$  in a  $\text{II}_1$  factor  $N$  we can form the algebra  $M_2(N)$  of  $2 \times 2$  matrices over  $N$ . We can construct a masa in  $M_2(N)$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a \in A, b \in B \right\},$$

which we denote  $A \oplus B$  — the direct sum of  $A$  and  $B$ . In [14] it is noted that if  $B$  is a unitary conjugate of  $A$ , then the Pukánszky invariant of  $A \oplus B$  can be determined from that of  $A$  (and hence  $B$ ). Indeed we have

$$\text{Puk}(A \oplus uAu^*) = \text{Puk}(A) \cup \{1\},$$

whenever  $u$  is a unitary in  $N$ . The initial motivation for the introduction of the mixed Pukánszky invariant was to aid in the study of the Pukánszky invariant of these direct sums since

$$\text{Puk}(A \oplus B) = \text{Puk}(A) \cup \text{Puk}(B) \cup \text{Puk}(A, B),$$

whenever  $A$  and  $B$  are masas in a  $\text{II}_1$  factor  $N$ . As we shall subsequently see, the Pukánszky invariant behaves badly with respect to the direct sum construction. In the next section we shall give Cartan masas  $A$  and  $B$  in the hyperfinite  $\text{II}_1$  factor such that  $\text{Puk}(A \oplus B) = \{1, n\}$  for any  $n \in \mathbb{N}_\infty$ , and given non-empty sets  $E, F, G \subset \mathbb{N}_\infty$  we shall construct, in Theorem 6.4, masas  $A$  and  $B$  in the hyperfinite  $\text{II}_1$  factor such that  $\text{Puk}(A) = E$ ,  $\text{Puk}(B) = F$  and  $\text{Puk}(A, B) = G$ . Hence it is not possible to make a more general statement about the Pukánszky invariant of a direct sum than

$$\text{Puk}(A \oplus B) \supset \text{Puk}(A) \cup \text{Puk}(B).$$

### 3 Mixed invariants of Cartan masas in $R$

In this section we shall construct large families of Cartan masas in the hyperfinite  $\text{II}_1$  factor, each masa will have Pukánszky invariant  $\{1\}$  by virtue of being Cartan [11, Section 3]. Our objective will be to control the mixed Pukánszky invariant of any two elements from the family. We start by constructing a family of three Cartan masas in the hyperfinite  $\text{II}_1$  factor and then use Lemma 2.4 to produce the desired result.

**Lemma 3.1.** *For each  $n \in \mathbb{N}_\infty$  there exists Cartan masas  $A, B, C$  in the hyperfinite  $\text{II}_1$  factor such that  $\text{Puk}(A, B) = \{n\}$  while  $\text{Puk}(A, C) = \text{Puk}(B, C) = \{1\}$ .*

We shall first establish Lemma 3.1 when  $n$  is finite. The lemma is immediate for  $n = 1$ , take  $A = B = C$  to be any Cartan masa in the hyperfinite  $\text{II}_1$  factor. Let  $n \geq 2$  be a fixed integer until further notice. Since any two

Cartan masas in the hyperfinite  $\text{II}_1$  factor are conjugate by an automorphism [2], we shall fix a Cartan masa  $A$  arising as the diagonals in an infinite tensor product and then construct  $B = \theta(A)$  and  $C = \phi(A)$  by exhibiting appropriate automorphisms  $\theta$  and  $\phi$  of  $R$ . Let  $M$  denote the  $n \times n$  matrices and  $D_0$  denote the diagonal  $n \times n$  matrices, a masa in  $M$ . Write  $(e_i)_{i=0}^{n-1}$  for the minimal projections of  $D_0$  so  $e_i$  has 1 in the  $(i, i)$ th entry and 0 elsewhere. Let

$$w = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

a unitary in  $M$ , which, in its action by conjugation, cyclically permutes the minimal projections of  $D_0$ . That is  $we_iw^* = e_{i-1}$  with the subtraction taken mod  $n$ . The abelian algebra generated by  $w$  is a masa  $D_1$  in  $M$ , which is orthogonal to  $D_0$  [10, Section 3]. Write  $(f_i)_{i=0}^{n-1}$  for the minimal projections of  $D_1$ . Define

$$v = \sum_{i=0}^{n-1} w^i \otimes f_i \quad (3.1)$$

a unitary in  $D_1 \otimes D_1 \subset M \otimes M$ .

We shall produce  $A, B$  and  $C$  in the hyperfinite  $\text{II}_1$  factor  $R$  realised as  $(\bigotimes_{r=1}^{\infty} M)''$ . Let  $A = (\bigotimes_{r=1}^{\infty} D_0)''$ . For each  $r$  consider the unitary  $u_r = 1^{\otimes(r-1)} \otimes v$ , which lies in  $M^{\otimes(r+1)} \subset R$ . All of these unitaries commute (as they lie in the masa  $(\bigotimes_{r=1}^{\infty} D_1)''$  in  $R$ ) and satisfy  $u_r^n = 1$ . We are able to define automorphisms

$$\theta = \lim_{r \rightarrow \infty} \text{Ad}(u_1 u_2 \dots u_r), \quad \phi = \lim_{r \rightarrow \infty} \text{Ad}(u_1 u_3 u_5 \dots u_{2r+1})$$

of  $R$  with the limit taken pointwise in  $\|\cdot\|_2$ . Convergence follows, since for  $x \in M^{\otimes r}$  we have  $u_s x u_s^* = x$  whenever  $s > r$  and such  $x$  are  $\|\cdot\|_2$ -dense in  $R$ . In this way  $\theta$  and  $\phi$  define  $*$ -isomorphisms of  $R$  into  $R$ . As  $\theta^n = I$  and  $\phi^n = I$  (since the  $u_r$ s commute and each  $u_r^n = 1$ ), we see that  $\theta$  and  $\phi$  are onto and so automorphisms of  $R$ . Define Cartan masas  $B = \theta(A)$  and  $C = \phi(A)$  in  $R$ . The calculations of  $\text{Puk}(A, C)$  and  $\text{Puk}(B, C)$  are straightforward.

**Lemma 3.2.** *With the notation above, we have  $\text{Puk}(A, C) = \text{Puk}(B, C) = \{1\}$ .*

*Proof.* We re-bracket the infinite tensor product defining  $R$  as

$$R = (M \otimes M) \overline{\otimes} (M \otimes M) \overline{\otimes} \dots$$

so that  $R$  is the infinite tensor product of copies of  $M \otimes M$ . Since  $u_{2r+1}$  lies in  $1^{\otimes 2r} \otimes (M \otimes M)$  we see that  $\phi$  factorises as  $\prod_{s=1}^{\infty} \text{Ad}(v)$  with respect to this decomposition. Lemma 2.4 then tells us that  $\text{Puk}(A, C)$  is the set product of infinitely many copies of  $\text{Puk}_{M \otimes M}(D_0 \otimes D_0, v(D_0 \otimes D_0)v^*)$ . Since  $D_0 \otimes D_0$  and  $v(D_0 \otimes D_0)v^*$  are masas in  $M_0 \otimes M_0$  a simple dimension check ensures that  $\text{Puk}_{M \otimes M}(D_0 \otimes D_0, v(D_0 \otimes D_0)v^*) = \{1\}$  and hence  $\text{Puk}(A, C) = \{1\}$ .

Observe that  $\text{Puk}(B, C) = \text{Puk}(\theta(A), \phi(A)) = \text{Puk}(\phi^{-1}\theta(A), A)$ . As all the  $u_r$  commute, we have

$$\phi^{-1} \circ \theta = \lim_{r \rightarrow \infty} \text{Ad}(u_2 u_4 \dots u_{2r})$$

with pointwise  $\|\cdot\|_2$  convergence. This time we re-bracket the tensor product defining  $R$  as

$$R = M \overline{\otimes} (M \otimes M) \overline{\otimes} (M \otimes M) \overline{\otimes} \dots,$$

and since  $u_{2r} = 1^{\otimes 2r-1} \otimes v \in 1 \otimes 1^{\otimes 2(r-1)} \otimes (M \otimes M)$ , we obtain  $\text{Puk}(B, C) = \{1\}$  in the same way.  $\square$   $\square$

The key tool in establishing that  $\text{Puk}(A, B) = \{n\}$  is the following calculation, which we shall use to produce  $n$  equivalent abelian projections for the commutant of the left-right action.

**Lemma 3.3.** *Use the notation preceding Lemma 3.2. For  $r = 0, 1, \dots, n-1$  let  $\xi_r$  denote  $f_r$  taken in the first copy of  $M$  in the tensor product making up  $R$ , thought of as a vector in  $L^2(R)$ . For any  $m \geq 0$ ,  $i_1, i_2, \dots, j_m, j_1, j_2, \dots, j_m = 0, 1, \dots, n-1$  and  $r, s = 0, 1, \dots, n-1$  we have*

$$\langle (e_{i_1} \otimes \dots \otimes e_{i_m}) \xi_r \theta(e_{j_1} \otimes \dots \otimes e_{j_m}), \xi_s \rangle_{L^2(R)} = \delta_{r,s} n^{-(2m+1)}. \quad (3.2)$$

*Proof.* We proceed by induction. When  $m = 0$ , (3.2) reduces to  $\langle \xi_r, \xi_s \rangle = \delta_{r,s} n^{-1}$ , which follows as  $\langle \xi_r, \xi_s \rangle = \text{tr}(f_r f_s^*)$  and  $(f_r)_{r=0}^{n-1}$  are the minimal projections of a masa in the  $n \times n$  matrices.

For  $m > 0$  observe that  $\theta(e_{j_1} \otimes \dots \otimes e_{j_m}) = u_1 \dots u_m (e_{j_1} \otimes \dots \otimes e_{j_m}) u_m^* \dots u_1^*$ . With the subtraction in the subscript taken mod  $n$ , we have

$$u_m(e_{j_1} \otimes \dots \otimes e_{j_m}) u_m^* = e_{j_1} \otimes \dots \otimes e_{j_{m-1}} \otimes \left( \sum_{k=0}^{n-1} e_{j_m-k} \otimes f_k \right)$$

from (3.1) and  $we_{j_m}w^* = e_{j_m-1}$ . Therefore

$$\begin{aligned}
& \langle (e_{i_1} \otimes \cdots \otimes e_{i_m}) \xi_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_m}), \xi_s \rangle \\
&= \left\langle (e_{i_1} \otimes \cdots \otimes e_{i_m}) \xi_r u_1 \dots u_{m-1} \left( e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes \sum_{k=0}^{n-1} e_{j_m-k} \otimes f_k \right) u_{m-1}^* \dots u_1^*, \xi_s \right\rangle \\
&= \text{tr} \left( \sum_{k=0}^{n-1} \left( (e_{i_1} \otimes \cdots \otimes e_{i_m}) f_r u_1 \dots u_{m-1} (e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_{j_m-k}) u_{m-1}^* \dots u_1^* f_s^* \right) \otimes f_k \right) \\
&= n^{-1} \text{tr} \left( (e_{i_1} \otimes \cdots \otimes e_{i_m}) f_r u_1 \dots u_{m-1} (e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes \sum_{k=0}^{n-1} e_{j_m-k}) u_{m-1}^* \dots u_1^* f_s^* \right) \\
&= n^{-1} \text{tr} \left( (e_{i_1} \otimes \cdots \otimes e_{i_m}) f_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1) f_s^* \right) \tag{3.3}
\end{aligned}$$

as the  $f_k$  in the third line is the only object appearing in the  $(m+1)$ -tensor position and  $\text{tr}$  is a product trace. This produces the factor  $n^{-1} = \text{tr}(f_k)$ . We obtain (3.3) as  $e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1$  lies in  $M^{\otimes(m-1)}$  so  $\theta(e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1) = u_1 \dots u_{m-1} (e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1) u_{m-1}^* \dots u_1^*$ .

Now  $\theta(f_r) = f_r$  for all  $r$  (since each  $u_m$  commutes with  $f_r$ ) and  $\theta$  is trace preserving. In this way we obtain

$$\begin{aligned}
& \langle (e_{i_1} \otimes \cdots \otimes e_{i_m}) \xi_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_m}), \xi_s \rangle \\
&= n^{-1} \text{tr} \left( \theta^{-1}(e_{i_1} \otimes \cdots \otimes e_{i_m}) f_r (e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1) f_s^* \right).
\end{aligned}$$

We now apply the same argument again giving us

$$\begin{aligned}
& \langle (e_{i_1} \otimes \cdots \otimes e_{i_m}) \xi_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_m}), \xi_s \rangle \\
&= n^{-2} \text{tr} \left( \theta^{-1}(e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes 1) f_r (e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1) f_s^* \right) \\
&= n^{-2} \text{tr} \left( (e_{i_1} \otimes \cdots \otimes e_{i_{m-1}}) f_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_{m-1}}) f_s^* \right) \\
&= n^{-2} \langle (e_{i_1} \otimes \cdots \otimes e_{i_{m-1}}) \xi_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_{m-1}}), \xi_s \rangle.
\end{aligned}$$

The lemma now follows by induction.  $\square$   $\square$

We can now complete the proof of Lemma 3.1.

*Proof.* Proof of Lemma 3.1. We continue to let  $n \geq 2$  be a fixed integer and let  $A$  and  $B$  be the masas introduced before Lemma 3.2. Let  $\mathcal{C}$  be the abelian algebra  $(A \cup JBJ)''$  in  $\mathbb{B}(L^2(R))$ . We continue to write  $\xi_r$  for  $f_r$  (in the first tensor position) thought of as a vector in  $L^2(R)$ . For each  $r$ , let  $P_r$  be the orthogonal projection in  $\mathbb{B}(L^2(R))$  onto  $\overline{\mathcal{C}\xi_r}$ , an abelian projection in  $\mathcal{C}'$ .

Since elements  $(e_{i_1} \otimes \cdots \otimes e_{i_m})f_r\theta(e_{j_1} \otimes \cdots \otimes e_{j_m})$ , where  $m \geq 0$  and  $i_1, \dots, i_m, j_1, \dots, j_m = 0, 1, \dots, n-1$ , have dense linear span in  $\overline{\mathcal{C}\xi_r}$ , Lemma 3.3 implies that  $P_r$  is orthogonal to  $P_s$  when  $r \neq s$ . Furthermore, for each  $m$ , the elements

$$(e_{i_1} \otimes \cdots \otimes e_{i_m})f_r\theta(e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes 1)$$

indexed by  $i_1, \dots, i_m, j_1, \dots, j_{m-1}, r = 0, 1, \dots, n-1$  are  $n^{2m}$  pairwise orthogonal non-zero elements of  $M^{\otimes m}$ , the  $n^m \times n^m$  matrices. Therefore,  $M^{\otimes m}$  is contained in the range of  $P_0 + P_1 + \cdots + P_{n-1}$  for each  $m$  so that  $\sum_{r=0}^{n-1} P_r = 1$ .

It remains to show that all the  $P_r$  are equivalent in  $\mathcal{C}'$ , from which it follows that  $\mathcal{C}'$  is homogeneous of type  $I_n$ . Given  $r \neq s$  we must define a partial isometry  $v_{r,s} \in \mathcal{C}'$  with  $v_{r,s}v_{r,s}^* = P_s$  and  $v_{r,s}^*v_{r,s} = P_r$ . Lemma 3.3 allows us to define  $v_{r,s}$  by extending the map  $\xi_r \mapsto \xi_s$  by  $(A, B)$ -modularity. More precisely define linear maps

$$v_{r,s}^{(m)} : \text{Span}(D_0^{\otimes m} f_r \theta(D_0^{\otimes m})) \rightarrow \text{Span}(D_0^{\otimes m} f_s \theta(D_0^{\otimes m}))$$

by extending

$$v_{r,s}^{(m)} \left( (e_{i_1} \otimes \cdots \otimes e_{i_m}) f_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_m}) \right) = (e_{i_1} \otimes \cdots \otimes e_{i_m}) f_s \theta(e_{j_1} \otimes \cdots \otimes e_{j_m})$$

by linearity. Lemma 3.3 shows that these maps preserve  $\|\cdot\|_2$  and that  $v_{r,s}^{(m+1)}$  extends  $v_{r,s}^{(m)}$ . Let  $v_{r,s}$  be the closure of the union of the  $v_{r,s}^{(m)}$ . This is patently a partial isometry in  $\mathcal{C}'$  with domain projection  $P_r$  and range projection  $P_s$ . Hence  $\text{Puk}(A, B) = \{n\}$  and combining this with Lemma 3.2 establishes Lemma 3.1 when  $n$  is finite.

When the  $n$  of Lemma 3.1 is  $\infty$  we take a tensor product. More precisely find Cartan masas  $A_0, B_0, C_0$  in the hyperfinite  $\text{II}_1$  factor  $R_0$  such that  $\text{Puk}(A_0, B_0) = \{2\}$  and  $\text{Puk}(A_0, C_0) = \text{Puk}(B_0, C_0) = \{1\}$ . Now form the hyperfinite  $\text{II}_1$  factor  $R$  by taking the infinite tensor product of copies of  $R_0$ . The Cartan masas  $A, B$  and  $C$  in  $R$  obtained from the infinite tensor product of copies of  $A_0, B_0$  and  $C_0$  have  $\text{Puk}(A, B) = \{\infty\}$ , and  $\text{Puk}(A, C) = \text{Puk}(B, C) = \{1\}$  by Lemma 2.4.  $\square$   $\square$

**Remark 3.4.** By fixing a Cartan masa  $D$  in a  $\text{II}_1$  factor  $N$  we could consider the map  $\theta \mapsto \text{Puk}(D, \theta(D))$ , which (by Proposition 2.3) induces a map on  $\text{Out}N$ . This map is not necessarily constant on outer conjugacy classes, as the automorphisms  $\theta$  and  $\phi$  of the hyperfinite  $\text{II}_1$  factor above have outer order  $n$  and obstruction to lifting 1 so are outer conjugate by [1].

Let us now give the main result of this section.

**Theorem 3.5.** *Let  $I$  be a countable set and let  $\Lambda$  be a symmetric matrix over  $\mathbb{N}_\infty$  indexed by  $I$ , with  $\Lambda_{i,i} = 1$  for all  $i \in I$ . There exist Cartan masas  $(D_i)_{i \in I}$  in the hyperfinite  $\text{II}_1$  factor such that  $\text{Puk}(D_i, D_j) = \{\Lambda_{i,j}\}$  for all  $i, j \in I$ .*

*Proof.* Let  $I$  and  $\Lambda$  be as in the statement of Theorem 3.5. For each unordered pair  $\{i, j\}$  of distinct elements of  $I$ , use Lemma 3.1 to find Cartan masas  $(D_r^{\{i,j\}})_{r \in I}$  in the copy of the hyperfinite  $\text{II}_1$  factor denoted  $R^{\{i,j\}}$  such that

$$\text{Puk}(D_r^{\{i,j\}}, D_s^{\{i,j\}}) = \begin{cases} \{\Lambda_{i,j}\} & \{r, s\} = \{i, j\} \\ \{1\} & \text{otherwise} \end{cases}.$$

This is achieved by taking  $D_i^{(i,j)} = A$ ,  $D_j^{(i,j)} = B$  and  $D_r^{(i,j)} = C$  for  $r \neq i, r \neq j$  where  $A, B, C$  are the masas resulting from taking  $n = \Lambda_{i,j}$  in Lemma 3.1. Now form the copy of the hyperfinite  $\text{II}_1$  factor  $R = \overline{\otimes}_{\{i,j\}} R^{\{i,j\}}$  and masas  $D_r = \overline{\otimes}_{\{i,j\}} D_r^{\{i,j\}}$  for  $r \in I$ . Lemma 2.4 ensures these masas have

$$\text{Puk}(D_i, D_j) = \{\Lambda_{i,j}\}$$

for all  $i, j \in I$ . □ □

We can immediately deduce the existence of masas with certain Pukánzsky invariants. The subsets below were first found in [7] using ergodic methods.

**Corollary 3.6.** *Let  $E$  be a finite subset of  $\mathbb{N}_\infty$  with  $1 \in E$ . Then there exists a masa in the hyperfinite  $\text{II}_1$  factor whose Pukánzsky invariant is  $E$ .*

*Proof.* If we work in the  $n \times n$  matrices  $M_n(R)$  over the hyperfinite  $\text{II}_1$  factor, and form the direct sum  $A = D_1 \oplus D_2 \oplus \cdots \oplus D_n$  of  $n$  Cartan masas, then

$$\text{Puk}(A) = \{1\} \cup \bigcup_{i < j} \text{Puk}(D_i, D_j).$$

The corollary then follows from Theorem 3.5 by choosing a large but finite  $I$  and appropriate values of  $\Lambda_{i,j}$  depending on the set  $E$ . □ □

All the pairs of Cartan masas we have produced have had a singleton for their mixed Pukánszky invariant. What are the possible values of  $\text{Puk}(A, B)$  when  $A$  and  $B$  are Cartan masas in a  $\text{II}_1$  factor?

## 4 The main construction

In this section we give a construction of masas in McDuff  $\text{II}_1$  factors, which we use to establish the main results of the paper in section 6. We need to introduce a not insubstantial amount of notation. Let  $N_0$  be a fixed separable McDuff  $\text{II}_1$  factor and for each  $r \in \mathbb{N}$ , let  $R^{(r)}$  be a copy of the hyperfinite  $\text{II}_1$  factor. Let  $N_r = N_0 \overline{\otimes} R^{(1)} \overline{\otimes} \dots \overline{\otimes} R^{(r)}$  so that with the inclusion map  $x \mapsto x \otimes 1_{R^{(r+1)}}$  we can regard  $N_r$  as a von Neumann subalgebra of  $N_{r+1}$ . We let  $N$  be the direct limit of this chain, so that

$$N = (N_0 \overline{\otimes} \bigotimes_{r=1}^{\infty} R^{(r)})''$$

acting on  $L^2(N_0) \otimes \bigotimes_{r=1}^{\infty} L^2(R^{(r)})$ . The  $\text{II}_1$  factor  $N$  is isomorphic to  $N_0$  and we shall regard all the  $N_r$  as subalgebras of  $N$ .

Whenever we have a masa  $D$  inside a  $\text{II}_1$  factor, we are able to use the isomorphism between  $D$  and  $L^\infty[0, 1]$  to choose families of projections  $e_i^{(m)}(D)$  in  $D$  for  $m \in \mathbb{N}$  and  $i = (i_1, \dots, i_m) \in \{0, 1\}^m$ , which satisfy:

1. For each  $m$  the  $2^m$  projections  $e_i^{(m)}(D)$  are pairwise orthogonal and each projection has trace  $2^{-m}$ ;
2. For each  $m$  and  $i = (i_1, \dots, i_m) \in \{0, 1\}^m$  we have

$$e_i^{(m)}(D) = e_{i \vee 0}^{(m+1)}(D) + e_{i \vee 1}^{(m+1)}(D),$$

where  $i \vee 0 = (i_1, \dots, i_m, 0)$  and  $i \vee 1 = (i_1, \dots, i_m, 1)$ ;

3. The projections  $e_i^{(m)}(D)$  generate  $D$ .

In the procedure that follows we shall assume that masas come with these projections when needed.

For  $m \in \mathbb{N}$  and  $r \geq 0$ , let  $I(r, m)$  denote the set of all  $i = (i^{(0)}, i^{(1)}, \dots, i^{(r)})$  where  $i^{(r-s)} = (i_1^{(r-s)}, i_2^{(r-s)}, \dots, i_{m+s}^{(r-s)}) \in \{0, 1\}^{m+s}$  is a sequence of zeros and ones of length  $m + s$ . In this way the last sequence,  $i^{(r)}$ , has length  $m$  and

each earlier sequence is one element longer than the following sequence. We have restriction maps from  $I(r, m)$  to  $I(r-1, m+1)$  obtained by forgetting about the last sequence  $i^{(r)}$ . Note that  $i^{(r-1)}$  has length  $m+1$  so that this restriction does lie in  $I(r-1, m+1)$ . We can also restrict by shortening the length of all the sequences. In full generality we have restriction maps from  $I(r, m)$  into  $I(s, l)$  whenever  $s \leq r$  and  $l \leq m+r-s$ . Given  $i \in I(r, m)$  and  $k \in I(s, l)$  (for  $s \leq r$  and  $l \leq m+r-s$ ) write  $i \geq k$  if the restriction of  $i$  to  $I(s, l)$  is precisely  $k$ . When  $i \in I(r, m)$  for some  $r$ , we write  $i|_s$  for the restriction of  $i$  to  $I(s, 1)$  for  $s \leq r$ . We take  $i|_{-1} = j|_{-1}$  as a convention for all  $i, j \in I(r, m)$ .

The inputs to our construction are a masa  $A_0$  in  $N_0$  and values  $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)} \in \mathbb{N}_\infty$  for all  $r = 0, 1, 2, \dots$  and  $i, j \in I(r, 1)$  with  $i \neq j$  and  $i|_{r-1} = j|_{r-1}$ . We regard these as fixed henceforth. For  $i \in I(0, m)$ , define  $f_i^{(0,m)} = e_{i(0)}^{(m)}(A_0)$ . Suppose inductively that we have produced masas  $A_s \subset N_s$  for each  $s \leq r$  and that, for each  $m \in \mathbb{N}$ , projections  $(f_i^{(s,m)})_{i \in I(s,m)}$  in  $A_s$  have been specified such that:

- (i) For each  $m \in \mathbb{N}$  and  $s \leq r$ , the  $|I(s, m)|$  projections  $(f_i^{(s,m)})_{i \in I(s,m)}$  are pairwise orthogonal and each has trace  $|I(m, s)|^{-1}$ ;
- (ii) For each  $m \in \mathbb{N}$ ,  $s \leq r$  and  $i \in I(s, m)$  we have

$$f_i^{(s,m)} = \sum_{\substack{j \in I(s,m+1) \\ j \geq i}} f_j^{(s,m+1)};$$

- (iii) For any  $s \leq t \leq r$  and  $i \in I(s, m+t-s)$  we have

$$f_i^{(s,m+t-s)} = \sum_{\substack{j \in I(t,m) \\ j \geq i}} f_j^{(t,m)},$$

noting that in this statement we regard the  $f^{(s,m+t-s)}$  as lying inside  $N_t$ ;

- (iv) For each  $s \leq r$  the projections  $\left\{ f_i^{(s,m)} \mid m \in \mathbb{N}, \quad i \in I(s, m) \right\}$  generate  $A_s$ .

Note that conditions (iii) and (iv) ensure that  $A_s \subset A_t$ .

To define  $A_{r+1}$ , use Theorem 3.5 to produce Cartan masas  $(D_i^{(r+1)})_{i \in I(r,1)}$  in  $R^{(r+1)}$  such that when  $i \neq j$  we have

$$\text{Puk} \left( D_i^{(r+1)}, D_j^{(r+1)} \right) = \begin{cases} \{\Lambda_{i,j}^{(r)}\} & i|_{r-1} = j|_{r-1} \\ \{1\} & \text{otherwise} \end{cases}. \quad (4.1)$$

Let  $A_{r+1}$  be given by

$$A_{r+1} = \bigoplus_{i \in I(r,1)} A_r f_i^{(r,1)} \otimes D_i^{(r+1)} \quad (4.2)$$

a masa in  $N_r \overline{\otimes} R^{(r+1)} = N_{r+1}$ , which has  $A_r \subset A_{r+1}$ . To complete the inductive construction we must define  $f_i^{(r+1,m)}$  for  $i \in I(r+1, m)$  in a manner which satisfies conditions (i) through (iv) above. Given  $m \in \mathbb{N}$  and  $i \in I(r+1, m)$ , let  $i'$  be the restriction of  $i$  to  $I(r, m+1)$  and recall that  $i|_r$  is the restriction of  $i$  to  $I(r, 1)$ . Now define

$$f_i^{(r+1,m)} = f_{i'}^{(r,m+1)} \otimes e_{i|_r}^{(m)}(D_{i|_r}^{(r+1)}). \quad (4.3)$$

Since  $f_{i'}^{(r,m+1)} \leq f_{i|_r}^{(r,1)}$ , this does define a projection in  $A_{r+1}$ . That the  $f_i^{(r+1,m)}$  satisfy the required conditions is routine. We give the details as Lemma 4.1 below for completeness.

**Lemma 4.1.** *The projections  $(f_i^{(r+1,m)})_{i \in I(r+1,m)}$  defined in (4.3) satisfy the conditions (i) through (iv) above.*

*Proof.* For  $m \in \mathbb{N}$  fixed, the projections  $(f_i^{(r+1,m)})_{i \in I(r+1,m)}$  are pairwise orthogonal and have trace  $|I(r+1, m)|^{-1}$  as the projections  $(f_{i'}^{(r,m+1)})_{i' \in I(r, m+1)}$  are pairwise orthogonal with trace  $|I(r, m+1)|^{-1}$  and the projections  $(e_j^{(m)}(D_{i|_r}^{(r+1)}))_{j \in \{0,1\}^m}$  are also pairwise orthogonal and each have trace  $2^{-m}$ . In this way the projections for  $A_{r+1}$  satisfy condition (i).

For condition (ii), fix  $i \in I(r+1, m)$  for some  $m \in \mathbb{N}$  and let  $i'$  be as in the definition of  $f_i^{(r+1,m)}$ . Now

$$\begin{aligned} f_i^{(r+1,m)} &= f_{i'}^{(r,m+1)} \otimes e_{i|_r}^{(m)}(D_{i|_r}^{(r+1)}) \\ &= \sum_{\substack{j' \in I(r, m+2) \\ j' \geq i'}} f_{j'}^{(r,m+2)} \otimes \left( e_{i|_r \vee 0}^{(m+1)}(D_{i|_r}^{(r+1)}) + e_{i|_r \vee 1}^{(m+1)}(D_{i|_r}^{(r+1)}) \right) \\ &= \sum_{\substack{j \in I(r+1, m+1) \\ j \geq i}} f_j^{(r+1, m+1)} \end{aligned}$$

from condition (ii) for the  $f_{i'}^{(r,m+1)}$  and the second condition in the definition of the  $e_k^{(m)}(D)$ . This is precisely condition (ii).

We only need to check condition (iii) when  $t = r+1$ , so take  $s \leq r$ ,  $m \in \mathbb{N}$  and  $i \in I(s, m+r+1-s)$ . By the inductive version of (iii) we have

$$f_i^{(s,m+r+1-s)} = \sum_{\substack{j \in I(r,m+1) \\ j \geq i}} f_j^{(r,m+1)}.$$

For each  $j \in I(r, m+1)$  with  $j \geq i$  we have

$$\begin{aligned} f_j^{(r,m+1)} \otimes 1_{R^{(r+1)}} &= f_j^{(r,m+1)} \otimes \sum_{j^{(r+1)} \in \{0,1\}^m} e_{j^{(r+1)}}^{(m)}(D_{j|_r}^{(r+1)}) \\ &= \sum_{\substack{k \in I(r+1,m) \\ k \geq j}} f_k^{(r+1,m+1)}, \end{aligned}$$

where  $j|_r$  is the restriction of  $j$  to  $I(r, 1)$ . Therefore,

$$f_i^{(s,m+r+1-s)} = \sum_{\substack{k \in I(r+1,m) \\ k \geq i}} f_k^{(r+1,m+1)},$$

which is condition (iii).

For  $j \in I(r, 1)$ , the projections  $f_k^{(r,m)}$  indexed by  $k \in I(r, m)$  with  $k \geq j$  generate the cut-down  $A_r f_j^{(r,1)}$ . Hence the projections  $f_i^{(r+1,m)}$ , for  $i \in I(r+1, m)$  with  $i \geq j$  generate  $A_r f_j^{(r,1)} \otimes D_j^{(r+1)}$ . In this way we see that the projections  $f_i^{(r+1,m)}$  for  $i \in I(r+1, m)$  generate  $A_{r+1}$ , which is condition (iv).  $\square$

This completes the inductive stage of the construction. We have masas  $A_r$  in  $N_r$  for each  $r$  such that  $A_r \otimes 1_{R^{(r+1)}} \subset A_{r+1}$ . We shall regard all these masas as subalgebras of the infinite tensor product  $\text{II}_1$  factor  $N$ , where they are no longer maximal abelian. Define  $A = (\bigcup_{r=0}^{\infty} A_r)''$ , an abelian subalgebra of  $R$ . For  $r \geq 0$  we have

$$A'_r \cap N = A_r \overline{\otimes} R^{(r+1)} \overline{\otimes} R^{(r+2)} \overline{\otimes} \dots$$

so that for  $x \in N_r \subset N$  we have  $\mathbb{E}_{A'_r \cap N}(x) = \mathbb{E}_{A_r}(x)$ , where  $\mathbb{E}_M$  denotes the unique trace-preserving conditional expectation onto the von Neumann subalgebra  $M$ . As  $A_r \subset A \subset A' \cap N \subset A'_r \cap N$  we obtain  $\mathbb{E}_A(x) = \mathbb{E}_{A' \cap N}(x)$  for any  $x \in \bigcup_{r=0}^{\infty} N_r$ . These  $x$  are weakly dense in  $N$  so  $A = A' \cap N$  is a masa in  $N$ , see [9, Lemma 2.1].

## 5 The Pukánszky invariant of $A$

Our objective here is to compute the Pukánszky invariant of the masas of section 4 in terms of the masa  $A_0$  and the specified values  $\Lambda_{i,j}^{(r)}$ . Following the usual convention, we shall write  $\mathcal{A}$  for the algebra  $(A \cup JAJ)''$ , an abelian subalgebra of  $\mathbb{B}(L^2(N))$ .

**Lemma 5.1.** *Let  $A$  be a masa produced by means of the construction described in section 4. Then*

$$\text{Puk}(A) = \bigcup_{r=0}^{\infty} \bigcup_{\substack{i,j \in I(r,1) \\ i \neq j \\ i|_{r-1}=j|_{r-1}}} \text{Type} \left( \mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J \right).$$

*Proof.* Fix  $s \geq 0, m \in \mathbb{N}$  and  $i \in I(s, m)$ . Let  $r = s + m - 1$ , so that condition (iii) gives

$$f_i^{(s,m)} = \sum_{\substack{j \in I(r,1) \\ j \geq i}} f_j^{(r,1)}.$$

Condition (iv) shows that the projections  $f_i^{(s,m)}$ , for  $m \in \mathbb{N}$  and  $i \in I(s, m)$ , generate  $A_s$ . Hence every  $A_s$  is contained in the abelian von Neumann algebra generated by all the  $f_i^{(r,1)}$  for  $i \in I(r, 1)$  and  $r \geq 0$ , so these projections generate  $A = (\bigcup_{s=1}^{\infty} A_s)''$ .

Writing  $B_r$  for the abelian von Neumann subalgebra of  $N$  generated by the projections  $(f_i^{(r,1)})_{i \in I(r,1)}$ , Lemma 2.1 of [9] shows us that

$$\lim_{r \rightarrow \infty} \left\| \mathbb{E}_{B'_r \cap N}(x) - \mathbb{E}_A(x) \right\|_2 = 0$$

for all  $x \in N$ , where  $\mathbb{E}_M$  denotes the trace-preserving conditional expectation onto the von Neumann subalgebra  $M$  of  $N$ . It is well known that  $\mathbb{E}_{B'_r \cap N} = \sum_{i \in I(r,1)} f_i^{(r,1)} J f_i^{(r,1)} J$  in this case, so

$$e_A = \lim_{r \rightarrow \infty} \sum_{i \in I(r,1)} f_i^{(r,1)} J f_i^{(r,1)} J,$$

with strong-operator convergence. Hence

$$1 - e_A = \sum_{r=0}^{\infty} \sum_{\substack{i,j \in I(r,1) \\ i \neq j \\ i|_{r-1}=j|_{r-1}}} f_i^{(r,1)} J f_j^{(r,1)} J$$

so the only contributions to the Pukánszky invariant of  $A$  come from the central cutdowns  $\mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J$  for  $r \geq 0$ ,  $i, j \in I(r, 1)$  with  $i \neq j$  and  $i|_{r-1} = j|_{r-1}$ .  $\square$   $\square$

For  $s \geq 0$ , write  $\mathcal{A}_s$  for the abelian von Neumann algebra  $(A_s \cup J A_s J)'' \subset \mathbb{B}(L^2(N_s))$ . For the rest of this section we shall denote operators in  $\mathbb{B}(L^2(N_s))$  with a superscript  $^{(s)}$ . Since

$$\mathbb{B}(L^2(N_{s+1})) = \mathbb{B}(L^2(N_s)) \overline{\otimes} \mathbb{B}(L^2(R^{(s+1)}))$$

we have  $T^{(s)} \otimes I_{L^2(R^{(s+1)})} \in \mathbb{B}(L^2(N_{s+1}))$  for all  $T^{(s)} \in \mathbb{B}(L^2(N_s))$ . We shall write  $T^{(s+1)}$  for this operator, and

$$T = T^{(s)} \otimes I_{L^2(R^{(s+1)})} \otimes I_{L^2(R^{(s+2)})} \otimes \dots$$

for this extension of  $T^{(s)}$  to  $L^2(N)$ . We refer to these operators as the canonical extensions of  $T^{(s)}$ . For  $T^{(s)} \in \mathcal{A}_s$ , we have  $T^{(s+1)} \in \mathcal{A}_{s+1}$  and  $T \in \mathcal{A}$ , since  $A_s \subset A_{s+1} \subset A$ . Let  $p_s$  denote the orthogonal projection from  $L^2(N)$  onto  $L^2(N_s)$ .

**Proposition 5.2.** *Let  $s \geq 0$  and  $T^{(s)} \in \mathbb{B}(L^2(N_s))$ . Then  $T^{(s)} \in \mathcal{A}'_s$  if and only if the extension  $T$  lies in  $\mathcal{A}'$ . Also  $p_s \mathcal{A}' p_s = \mathcal{A}'_s$ .*

*Proof.* Let  $T \in \mathbb{B}(L^2(N))$  lie in  $\mathcal{A}'$ . For each  $s$  and  $x \in A_s$ , we have  $p_s x p_s = x p_s = p_s x$  and  $p_s J x J p_s = J x J p_s = p_s J x J$ . Then  $p_s T p_s$  commutes with both  $x$  and  $J x J$  and hence lies in  $\mathcal{A}'_s$ . This gives  $p_s \mathcal{A}' p_s \subset \mathcal{A}'_s$  and shows that if  $T$  is the canonical extension of some  $T^{(s)} \in \mathbb{B}(L^2(N_s))$ , then  $T^{(s)} \in \mathcal{A}'_s$ .

For the converse, consider  $T^{(s)} \in \mathcal{A}'_s$  and take  $x \in A_{s+1}$  so that

$$x = \sum_{i \in I(s,1)} x_i f_i^{(s,1)} \otimes y_i$$

for some  $x_i \in A_s$  and  $y_i \in D_i^{(s+1)}$  by the inductive definition of  $A_{s+1}$  in equation (4.2). Then  $T^{(s+1)}$  commutes with  $x$  since  $T^{(s)}$  commutes with each  $x_i f_i^{(s,1)}$ . Similarly  $T^{(s+1)}$  commutes with  $J x J$ , so  $T^{(s+1)} \in \mathcal{A}'_{s+1}$ . Proceeding by induction, we see that  $T^{(r)} \in \mathcal{A}'_r$  for all  $r \geq s$ . Hence, the canonical extension  $T$  commutes with  $x$  and  $J x J$  for all  $x \in \bigcup_{r=0}^{\infty} A_r$  and these elements are weakly dense in  $\mathcal{A}$ , so  $T \in \mathcal{A}'$ . For  $T^{(s)} \in \mathbb{B}(L^2(N_s))$  the canonical extension  $T$  has  $p_s T p_s = T^{(s)}$ , so  $\mathcal{A}'_s \subset p_s \mathcal{A}' p_s$ .  $\square$   $\square$

Our objective is to determine the type decomposition of the  $\mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J$  appearing in Lemma 5.1. For  $r \geq 0$  and  $i \in I(r, 1)$ , the inductive definition (4.3) ensures that

$$f_i^{(r,1)} = e_{i(0)}^{(r+1)}(A_0) \otimes e_{i(1)}^{(r)}(D_{i|_0}^{(1)}) \otimes \cdots \otimes e_{i(r)}^{(1)}(D_{i|_{r-1}}^{(r)})$$

recalling that  $i|_s$  is the restriction of  $i$  to  $I(s, 1)$ .

**Lemma 5.3.** *Let  $r \geq 0$  and  $i, j \in I(r, 1)$  have  $i \neq j$  and  $i|_{r-1} = j|_{r-1}$ . Let  $Q^{(0)} \in \mathcal{A}_0 e_{i(0)}^{(r+1)}(A_0) J e_{j(0)}^{(r+1)}(A_0) J$  be a non-zero projection such that  $\mathcal{A}'_0 Q^{(0)}$  is homogeneous of type  $I_m$  for some  $m \in \mathbb{N}_\infty$ . Then, writing  $Q$  for the canonical extension of  $Q^{(0)}$  to  $L^2(N)$ ,  $\mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J Q$  is homogeneous of type  $I_{m\Lambda_{i,j}^{(r)}}$ .*

*Proof.* Fix  $m \in \mathbb{N}_\infty$  and  $Q^{(0)} \neq 0$  as in the statement of the Lemma. Observe that

$$\begin{aligned} A_{r+1} f_i^{(r,1)} &= A^{(r)} f_i^{(r,1)} \overline{\otimes} D_i^{(r+1)} \\ &= A_0 e_{i(0)}^{(r+1)}(A_0) \overline{\otimes} D_{i|_0}^{(1)} e_{i(1)}^{(r)}(D_{i|_0}^{(1)}) \overline{\otimes} \cdots \overline{\otimes} D_{i|_{r-1}}^{(r)} e_{i(r)}^{(1)}(D_{i|_{r-1}}^{(r)}) \overline{\otimes} D_i^{(r+1)} \end{aligned}$$

so that

$$\begin{aligned} &\mathcal{A}_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)} \\ &= \mathcal{A}_0 Q^{(0)} \overline{\otimes} (D_{i|_0}^{(1)} \cup J D_{i|_0}^{(1)} J)'' e_{i(1)}^{(r)}(D_{i(0)}^{(1)}) J e_{j(1)}^{(r)}(D_{i(0)}^{(1)}) J \\ &\quad \overline{\otimes} \cdots \overline{\otimes} (D_{i|_{r-1}}^{(r)} \cup J D_{i|_{r-1}}^{(r)} J)'' e_{i(r)}^{(1)}(D_{i|_{r-1}}^{(r)}) J e_{j(r)}^{(1)}(D_{i|_{r-1}}^{(r)}) J \overline{\otimes} (D_i^{(r+1)} \cup J D_j^{(r+1)} J)'', \end{aligned}$$

using  $i|_s = j|_s$  for  $s = 0, \dots, r-1$ . We are also abusing notation by writing  $J$  for the modular conjugation operator regardless of the space on which it operates. Taking commutants gives

$$\begin{aligned} &\mathcal{A}'_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)} \\ &= \mathcal{A}'_0 Q^{(0)} \overline{\otimes} (D_{i|_0}^{(1)} \cup J D_{i|_0}^{(1)} J)' e_{i(1)}^{(r)}(D_{i(0)}^{(1)}) J e_{j(1)}^{(r)}(D_{i(0)}^{(1)}) J \\ &\quad \overline{\otimes} \cdots \overline{\otimes} (D_{i|_{r-1}}^{(r)} \cup J D_{i|_{r-1}}^{(r)} J)' e_{i(r)}^{(1)}(D_{i|_{r-1}}^{(r)}) J e_{j(r)}^{(1)}(D_{i|_{r-1}}^{(r)}) J \overline{\otimes} (D_i^{(r+1)} \cup J D_j^{(r+1)} J)'. \end{aligned}$$

For  $s \leq r$ , each  $(D_{i|_{s-1}}^{(s)} \cup J D_{i|_{s-1}}^{(s)} J)''$  is maximal abelian in  $\mathbb{B}(L^2(R^{(s)}))$  since  $D_{i|_{s-1}}^{(s)}$  is a Cartan masa so has Pukánszky invariant  $\{1\}$ . The masas  $D_k^{(r+1)}$  where defined in (4.1) so that  $(D_i^{(r+1)} \cup J D_j^{(r+1)} J)'$  is homogeneous of type

$I_{\Lambda_{i,j}^{(r)}}$ . We learn that  $\mathcal{A}'_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)}$  is homogeneous of type  $I_{m'}$ , where  $m' = m \Lambda_{i,j}^{(r)}$ .

Find a family of pairwise orthogonal projections  $(Q_q^{(r+1)})_{0 \leq q < m'}$  with sum  $Q^{(r+1)}$  and which are equivalent abelian projections in  $\mathcal{A}'_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)}$ . The canonical extensions  $(Q_q)_{0 \leq q < m'}$  to  $L^2(N)$  form a family of pairwise orthogonal projections in  $\mathcal{A}'Q$  (by Proposition 5.2) with sum  $Q$ . These projections are equivalent in  $\mathcal{A}'Q$  as if  $V^{(r+1)}$  is a partial isometry in  $\mathcal{A}'_r Q^{(r+1)}$  with  $V^{(r+1)} V^{(r+1)*} = Q_q$  and  $V^{(r+1)*} V^{(r+1)} = Q_{q'}$ , then Proposition 5.2 ensures that the canonical extension  $V$  lies in  $\mathcal{A}'$ . It is immediate that  $VV^* = Q_q$  and  $V^*V = Q_{q'}$ . We shall show that these projections are abelian projections in  $\mathcal{A}'$ . It will then follow that  $\mathcal{A}'Q$  is homogeneous of type  $I_{m'}$ .

For  $s \geq r+1$  and  $k, l \in I(s, 1)$  with  $k \geq i$  and  $l \geq j$ , we have

$$A_{s+1} f_k^{(s,1)} = A_s f_k^{(s,1)} \overline{\otimes} D_k^{(s+1)}$$

so that

$$\mathcal{A}_{s+1}(f_k^{(s,1)} J f_l^{(s,1)} J) Q^{(s+1)} \cong \mathcal{A}_s(f_k^{(s,1)} J f_l^{(s,1)} J) Q^{(s)} \overline{\otimes} (D_k^{(s+1)} \cup J D_l^{(s+1)} J)'.$$

Again we take commutants to obtain

$$\mathcal{A}'_{s+1}(f_k^{(s,1)} J f_l^{(s,1)} J) Q^{(s+1)} \cong \mathcal{A}'_s(f_k^{(s,1)} J f_l^{(s,1)} J) Q^{(s)} \overline{\otimes} (D_k^{(s+1)} \cup J D_l^{(s+1)} J)'.$$

Since  $i \neq j$  it is not possible for  $k|_s$  to equal  $l|_s$ , so (4.1) shows us that  $(D_k^{(s+1)} \cup J D_l^{(s+1)} J)'$  is abelian. Therefore, if  $Q_q^{(s)} f_k^{(s,1)} J f_l^{(s,1)} J$  (some  $q = 1, \dots, m'$ ) is an abelian projection in  $\mathcal{A}'_s$ , then  $Q_q^{(s+1)} f_k^{(s+1,1)} J f_l^{(s+1,1)} J$  is abelian in  $\mathcal{A}'_{s+1}$ . The projections  $f_k^{(s,1)} J f_l^{(s,1)} J$  are central and satisfy

$$\sum_{\substack{k, l \in I(s, 1) \\ k \geq i \\ l \geq j}} f_k^{(s,1)} J f_l^{(s,1)} J = f_i^{(r,1)} J f_j^{(r,1)} J.$$

By induction and summing over all  $k \geq i$  and  $l \geq j$ , we learn that  $(Q_q^{(s)})_{0 \leq q < m'}$  form a family of equivalent abelian projections in  $\mathcal{A}'Q^{(s)}$  with sum  $s$  for every  $s \geq r+1$ .

For  $s \geq r+1$  and each  $q$ , the algebras  $\mathcal{A}'_s Q_q^{(s)} = p_s \mathcal{A}' Q_q p_s$  are abelian. Since the projections  $p_s$  tend strongly to the identity, we see that each  $\mathcal{A}' Q_q$  is abelian too.  $\square$   $\square$

We can now describe the Pukánszky invariant of the masas in section 4.

**Theorem 5.4.** *Let  $A$  be a masa in a separable McDuff  $II_1$  factor produced via the construction of section 4. That is we are given a masa  $A_0 \subset N_0$  and values  $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)} \in \mathbb{N}_\infty$  for  $r \geq 0$ ,  $i, j \in I(r, 1)$  with  $i \neq j$  and  $i|_{r-1} = j|_{r-1}$ . Then*

$$\text{Puk}(A) = \bigcup_{r=0}^{\infty} \bigcup_{\substack{i,j \in I(r,1) \\ i \neq j \\ i|_{r-1} = j|_{r-1}}} \Lambda_{i,j}^{(r)} \cdot \text{Type} \left( \mathcal{A}'_0 e_{i(0)}^{(r+1)}(A_0) J e_{j(0)}^{(r+1)}(A_0) J \right). \quad (5.1)$$

*Proof.* For  $r \geq 0$ ,  $i, j \in I(r, 1)$  with  $i \neq j$  and  $i|_{r-1} = j|_{r-1}$ , it follows from Lemma 5.3 that

$$\text{Type} \left( \mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J \right) = \Lambda_{i,j}^{(r)} \cdot \text{Type} \left( \mathcal{A}'_0 e_{i(0)}^{(r+1)}(A_0) \cup J e_{j(0)}^{(r+1)}(A_0) J \right).$$

The theorem then follows from Lemma 5.1.  $\square$   $\square$

## 6 Main results

We start by applying Theorem 5.4 when  $\text{Puk}(A_0)$  is a singleton.

**Theorem 6.1.** *For  $n \in \mathbb{N}$ , suppose that  $N_0$  is a separable McDuff  $II_1$  factor containing a masa with Pukánszky invariant  $\{n\}$ . For every non-empty set  $E \subset \mathbb{N}_\infty$ , there exists a masa  $A$  in  $N_0$  with  $\text{Puk}(A) = \{n\} \cdot E$ .*

*Proof.* Let  $A_0$  be a masa in  $N_0$  with  $\text{Puk}(A) = \{n\}$  and choose the values  $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)}$  for  $r \geq 0$  and  $i, j \in I(r, 1)$  with  $i \neq j$  and  $i|_{r-1} = j|_{r-1}$  so that

$$E = \left\{ \Lambda_{i,j}^{(r)} \mid r \geq 0, \quad i, j \in I(r, 1), \quad i \neq j, \quad i|_{r-1} = j|_{r-1} \right\}.$$

The resulting masa  $A$  in  $N \cong N_0$  produced by the main construction has Pukánszky invariant  $\{n\} \cdot E$  by Theorem 5.4.  $\square$   $\square$

Since Cartan masas have Pukánszky invariant  $\{1\}$ , we obtain the following Corollary immediately.

**Corollary 6.2.** *Let  $N$  be a McDuff  $II_1$  factor containing a simple masa, for example a Cartan masa. Every non-empty subset of  $\mathbb{N}_\infty$  arises as the Pukánszky invariant of a masa in  $N$ .*

A little more care enables us to address the question of the range of the Pukánszky invariant on singular masas in the hyperfinite  $\text{II}_1$  factor and other McDuff  $\text{II}_1$  factors containing a simple singular masa. Pukánszky's original work [12] exhibits a simple singular masa in the hyperfinite  $\text{II}_1$  factor.

**Corollary 6.3.** *Let  $N$  be a separable McDuff factor containing a simple singular masa, such as the hyperfinite  $\text{II}_1$  factor. Given any non-empty  $E \subset \mathbb{N}_\infty$  there is a singular masa  $A$  in  $N$  with  $\text{Puk}(A) = E$ .*

*Proof.* If  $1 \notin E$ , a masa in  $N$  with Pukánszky invariant  $E$  is automatically singular by [11, Remark 3.4]. We have already produced these masas in Corollary 6.2. The hypothesis ensures us a simple singular masa in  $N$ . For the remaining case of some  $E \neq \{1\}$  with  $1 \in E$ , let  $A_1$  be a singular masa in  $N$  with  $\text{Puk}_{N_1}(A_1) = \{1\}$  and  $A_2$  be a singular masa in the hyperfinite  $\text{II}_1$  factor  $R$  with  $\text{Puk}_R(A_2) = E \setminus \{1\}$ . Then  $A = A_1 \overline{\otimes} A_2$  is a masa in  $N \overline{\otimes} R \cong N$ . Lemma 2.1 of [14] ensures that

$$\text{Puk}(A) = \{1\} \cup (E \setminus \{1\}) \cup 1 \cdot (E \setminus \{1\}) = E.$$

The singularity of  $A$  is Corollary 2.4 of [15]. □ □

Next we justify the claims made at the end of section 2.

**Theorem 6.4.** *Let  $E, F, G \subset \mathbb{N}_\infty$  be non-empty. Then there exist masas  $B$  and  $C$  in the hyperfinite  $\text{II}_1$  factor with  $\text{Puk}(B) = E$ ,  $\text{Puk}(C) = F$  and  $\text{Puk}(B, C) = G$ .*

*Proof.* Let  $R_0$  be a copy of the hyperfinite  $\text{II}_1$  factor and  $A_0$  a Cartan masa in  $R_0$ . An element  $k$  of  $I(0, 1)$  is of the form  $(k^{(0)})$  where  $k^{(0)}$  is a 1-tuple — either 0 or 1. Write  $\mathbf{0}$  and  $\mathbf{1}$  for these two elements and let  $e_0 = f_{\mathbf{0}}^{(1)}$  and  $e_1 = f_{\mathbf{1}}^{(1)}$  so that  $e_0$  and  $e_1$  are orthogonal projections in  $A$  with  $\text{tr}(e_0) = \text{tr}(e_1) = 1/2$ . Choose the  $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)}$  such that:

$$\begin{aligned} E &= \left\{ \Lambda_{i,j}^{(r)} \mid r \geq 1, \quad i, j \in I(r, 1), i \neq j, i|_{r-1} = j|_{r-1}, i, j \geq \mathbf{0} \right\}, \\ F &= \left\{ \Lambda_{i,j}^{(r)} \mid r \geq 1, \quad i, j \in I(r, 1), i \neq j, i|_{r-1} = j|_{r-1}, i, j \geq \mathbf{1} \right\}, \\ G &= \left\{ \Lambda_{i,j}^{(r)} \mid r \geq 0, \quad i, j \in I(r, 1), i \neq j, i|_{r-1} = j|_{r-1}, i \geq \mathbf{0}, j \geq \mathbf{1} \right\} \\ &= \left\{ \Lambda_{i,j}^{(r)} \mid r \geq 0, \quad i, j \in I(r, 1), i \neq j, i|_{r-1} = j|_{r-1}, i \geq \mathbf{1}, j \geq \mathbf{0} \right\}. \end{aligned}$$

For  $r, s = 0, 1$ , let  $Q_{r,s} = (1 - e_A)e_r J e_s J$  a projection in  $\mathcal{A}$ . Now Lemma 5.3 and Lemma 5.1 ensure that  $\mathcal{A}'Q_{0,0}$  has a non-zero  $I_m$  cutdown if and only if  $m \in E$ ,  $\mathcal{A}'Q_{1,1}$  has a non-zero  $I_m$  cutdown if and only if  $m \in F$ ,  $\mathcal{A}'(Q_{0,1} + Q_{1,0})$  has a non-zero  $I_m$  cutdown if and only if  $m \in G$ .

We now regard  $A$  as a direct sum. Consider the copy of the hyperfinite  $\text{II}_1$  factor  $S = e_0 R e_0$  so that choosing a partial isometry  $v \in R$  with  $v^*v = e_0$  and  $vv^* = e_1$  gives rise to an isomorphism between  $R$  and  $M_2(S)$  — the  $2 \times 2$  matrices over  $S$ . Define masas in  $S$  by  $B = Ae_0$  and  $C = v^*(Ae_1)v$ . The discussion above ensures that  $\text{Puk}(B) = E$ ,  $\text{Puk}(C) = F$  and  $\text{Puk}(B, C) = G$ . Note that  $\text{Puk}(B, C)$  is independent of  $v$  by Proposition 2.3.  $\square \quad \square$

**Remark 6.5.** If  $E \subset \mathbb{N}_\infty$  contains at least two elements then we can modify the construction in section 4 to produce uncountably many pairwise non-conjugate masas in the hyperfinite  $\text{II}_1$  factor  $R$  each with Pukánszky invariant  $E$ . The idea is to control the supremum of the trace of a projection in the masa  $A$  such that  $\text{Puk}_{eRe}(Ae) = \{n\}$  for some fixed  $n \in E$ . For each  $t \in (0, 1)$ , we can produce masas  $A$  in  $R$  and a projection  $e \in A$  with  $\text{tr}(e) = t$  such that (with the intuitive diagrams of the introduction) the multiplicity structure of  $\mathcal{A}$  is represented by Figure 5, with 1 down the diagonal and  $E \setminus \{n\}$  occurring in the unmarked areas. All these masas must be pairwise non-conjugate.

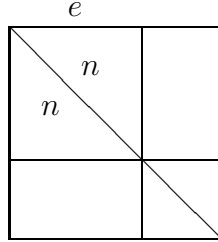


Figure 5: The multiplicity structure of  $\mathcal{A}$ .

No modifications are required to obtain any dyadic rational for  $t$ , we follow Theorem 6.4 to control the multiplicity structure of  $\mathcal{A}$ . For general  $t$  we can approximate the required structure using dyadic rationals, leaving the area we are unable to handle at each stage with multiplicity 1 so it can be adjusted at a subsequent stage.

**Remark 6.6.** For a masa  $A$  in a property  $\Gamma$ -factor  $N$ , the property that  $A$  contains non-trivial centralising sequences for  $N$  has been used to distinguish between non-conjugate masas, see for example [5, 7, 13]. We can easily adjust the construction of section 4 to ensure that all the masas produced have this property. Suppose that we identify each  $R^{(r)}$  with  $R^{(r)} \overline{\otimes} R^{(r)}$  and we replace the masas  $D_i^{(r)}$  in  $R^{(r)}$  by  $D_i^{(r)} \overline{\otimes} E^{(r)}$  where  $E^{(r)}$  is a fixed Cartan masa in  $R^{(r)}$ . By Lemma 2.4 this does not alter the mixed Pukánszky invariants of the family, so the Pukánszky invariant of the masa resulting from the construction remains unchanged. This masa now contains non-trivial centralising sequences for  $N$ . By way of contrast, the examples in [14, 4] arise from inclusions  $H \subset G$  of an abelian group inside a discrete I.C.C. group  $G$  with  $gHg^{-1} \cap H = \{1\}$  for all  $g \in G \setminus H$ . The resulting masa  $\mathcal{L}(H)$  can not contain non-trivial centralising sequences for the  $\text{II}_1$  factor  $\mathcal{L}(G)$ , [10].

Very recently Ozawa and Popa have shown that not every McDuff  $\text{II}_1$  factor contains a Cartan masa. Indeed in [8] they show that there are no Cartan masas in  $\mathcal{LF}_2 \overline{\otimes} R$ . It is not known whether every McDuff factor must contain a simple masa (one with Pukánszky invariant  $\{1\}$ ) or a masa whose Pukánszky invariant is a finite subset of  $\mathbb{N}$ . We can however obtain subsets containing  $\infty$  as Pukánszky invariants of masas in a general separable McDuff  $\text{II}_1$  factor.

**Theorem 6.7.** *Let  $N$  be a separable McDuff  $\text{II}_1$  factor. For every set  $E \subset \mathbb{N}_\infty$  with  $\infty \in E$  there is a singular masa  $B$  in  $N$  with  $\text{Puk}(B) = E$ .*

*Proof.* Taking all the  $\Lambda_{i,j}^{(r)} = \infty$ , gives us a masa  $A$  in  $N$  with  $\text{Puk}(A) = \{\infty\}$  by Theorem 5.4 (regardless of the masa  $A_0$ ). Now use the isomorphism  $N \cong N \overline{\otimes} R$ , where  $R$  is the hyperfinite  $\text{II}_1$  factor. Let  $B = A \overline{\otimes} A_1$ , where  $A_1$  is a singular masa in  $R$  with  $\text{Puk}_R(A_1) = E$ . Lemma 2.1 of [14] gives

$$\text{Puk}(B) = \{\infty\} \cup E \cup \{\infty\} \cdot E = E. \quad \square$$

□

In particular every separable McDuff  $\text{II}_1$  factor contains uncountably many pairwise non-conjugate singular masas.

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